

EXAMPLES OF NON-KÄHLER HAMILTONIAN TORUS ACTIONS

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ABSTRACT. An important question with a rich history is the extent to which the symplectic category is larger than the Kähler category. Many interesting examples of non-Kähler symplectic manifolds have been constructed [T] [M] [G]. However, sufficiently large symmetries can force a symplectic manifold to be Kähler [D] [Kn]. In this paper, we solve several outstanding problems by constructing the first symplectic manifold with large non-trivial symmetries which does not admit an invariant Kähler structure. The proof that it is not Kähler is based on the Atiyah-Guillemin-Sternberg convexity theorem [At] [GS]. Using the ideas of this paper, C. Woodward shows that even the symplectic analogue of spherical varieties need not be Kähler [W].

1. INTRODUCTION

A **symplectic manifold** is a manifold M with a nondegenerate closed two form ω . A complex structure $J : TM \rightarrow TM$ is **compatible** with ω if $g(X, Y) := \omega(J(X), Y)$ is a Riemannian metric. A **Kähler structure** is a symplectic form ω and a compatible complex structure J on M . Many theorems are much easier to prove for Kähler manifolds than for symplectic manifolds. Indeed, some are only known to be true in the Kähler case, and others are known to be true only in the Kähler case. Therefore, an important question with a rich history is the extent to which the symplectic category is larger than the Kähler category.

Many interesting examples of non-Kähler symplectic manifolds have been constructed. For example, in 1971 Thurston constructed a compact four dimensional symplectic manifold which is not Kähler [T]. In 1984, D. McDuff used Thurston's example and symplectic blowups to construct a compact ten dimensional *simply connected* non-Kähler symplectic manifold [M]. There are two ways to show that these manifolds do not admit Kähler structures; they each have an odd Betti number which is odd, and their cohomology does not have the hard Lefschetz property. In 1994, among many other examples, R. Gompf constructed a symplectic form on a Gompf-Mrowka manifold, a four dimensional simply connected symplectic manifold which does not admit any complex structure [G] [GM].

Let T be a torus which acts on a manifold M , preserving a symplectic form ω . The action is **Hamiltonian** if there exists a **moment map** $\phi : M \rightarrow \mathfrak{t}^*$ such that $i_{\xi_M} \omega = -d\langle \phi, \xi \rangle$ for all $\xi \in \mathfrak{t}$, where ξ_M is the induced vector field on M . For mathematicians who study Hamiltonian actions, the important question is to what extent do these symmetries force symplectic manifolds to admit invariant Kähler structures.

On the one hand, it is easy to find a non-Kähler symplectic manifold with a Hamiltonian group action. Let M be a symplectic manifold which does not admit a Kähler structure. The action of S^1 on $M \times \mathbb{CP}^1$ given by $\lambda \cdot (m, [x, y]) = (m, [\lambda x, y])$ is Hamiltonian, and M' does not admit an invariant Kähler structure. However,

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this example is not very interesting. The components of the fixed point set and the reduced spaces are themselves symplectic manifolds which do not admit Kähler structures.

More interestingly, E. Lerman used McDuff's example to construct a compact twelve dimensional simply connected symplectic manifold and a Hamiltonian circle action on it with an isolated fixed point [L]. The manifold is not Kähler because it has an odd dimensional odd Betti number. In this example, some of the fixed point sets and some of the reduced spaces are themselves non-Kähler.

On the other hand, sufficiently large symmetries can force a symplectic manifold to be Kähler. Let a torus T act effectively on a compact symplectic manifold (M, ω) in a Hamiltonian fashion. T. Delzant shows that if the dimension of T is half the dimension of M , then (M, ω) admits a T invariant compatible complex structure [D]. Similarly, if M is four dimensional and T is one dimensional, then Y. Karshon shows that (M, ω) admits a T invariant compatible complex structure [Ka]. (See also [Au] and [AH]).

In this paper, we give the first symplectic manifold with large non-trivial symmetries which does not admit an invariant Kähler structure. We form a six dimensional compact symplectic manifold with an effective Hamiltonian two dimensional torus action. Using Atiyah-Guillemin-Sternberg convexity theorem [At] [GS], we develop a new criterion for showing that a symplectic manifold does not admit an invariant complex structure, and use this to show that our example does not admit any invariant Kähler structure. Moreover, using these techniques it is possible to find many other symplectic manifolds with similar properties.

This example solves several outstanding existence problems. For instance, it shows that a symplectic manifold with a Hamiltonian torus action need not admit an invariant Kähler structure even if all the reduced spaces are Kähler. It is the first known example with isolated fixed points. This example is sharp in the sense that it is not possible to find either an example with a lower dimensional symplectic manifold, or an example with a larger Hamiltonian torus action (relative to the symplectic manifold). Additionally, this manifold is simply connected, and the action is quasi-free.

However, the most important implication of this paper is that the category of symplectic manifolds with Hamiltonian torus actions is much richer than the category of Kähler manifolds with compatible Hamiltonian torus actions. In recent years there has been a degree of convergence between the study of symplectic manifolds with Hamiltonian group actions, and the study of algebraic varieties with algebraic group actions. Some mathematicians had feared – or hoped – that traditional Hamiltonian techniques might be rendered obsolete. After all, more powerful techniques are often available in algebraic geometry. However, I believe that this family of examples shows that Kähler manifolds, far from being the rule, are merely an interesting special case – even among highly symmetric symplectic manifolds.

Using the ideas of this paper, C. Woodward shows that even the symplectic analogue of spherical varieties need not be Kähler [W]. He finds an alternate construction for my example and for several others, using symplectic cuts. In so doing, he shows that they in fact admit a multiplicity free Hamiltonian $U(2)$ action. F. Knop has independently constructed other symplectic manifolds which admit a multiplicity free Hamiltonian G action, where G is a compact nonabelian group. He shows that they do not admit compatible G invariant complex structures by using the classification of Kähler manifolds which admit multiplicity free Hamiltonian actions [Kn].

In Section 2, we define the x-ray of a compact symplectic manifold with a Hamiltonian torus action.

In Section 3, we prove that if the x-ray of a symplectic manifold with a Hamiltonian torus action does not satisfy the extension criterion, then the symplectic manifold does not admit a compatible invariant complex structure.

In Section 4, we glue together pieces of two Kähler manifolds to construct a symplectic manifold with a Hamiltonian torus action. Because its x-ray does not satisfy the extension criterion, it does not admit an invariant compatible complex structure.

In Section 5, we show that the manifold which we constructed in Section 4 does not admit any T invariant Kähler structure.

In Section 6, we describe how to construct more symplectic manifolds with Hamiltonian torus actions which do not admit compatible invariant complex structures.

1.1. Acknowledgements. I am happy to have this opportunity to thank Yael Karshon for all that she taught me about symplectic geometry. In particular, my ideas about x-rays, two torus actions on six manifolds, and symplectic toric manifolds all grew out of joint projects with her, and made this paper possible.

Similarly, I would like to thank Chris Woodward for his enthusiasm and for his invaluable help with many aspects of this project.

I am grateful to Eugene Lerman for introducing me to this problem and for offering me mathematical and practical advice.

2. THE X-RAY

In this section, we define the x-ray of a compact symplectic manifold with a Hamiltonian torus action. In the rest of the paper, we will use the x-ray both to construct an example, and to show that it does not admit an invariant Kähler structure.

Let a torus T act on a compact symplectic manifold M with moment map $\phi : M \rightarrow \mathfrak{t}^*$. For each subgroup $K \subset T$, let M^K be the set of points fixed by K , and let \mathcal{X}_K be the set of connected components of M^K . The **closed orbit type stratification** of M is the set $\mathcal{X} = \cup_{K \subset T} \mathcal{X}_K$; this set is partial ordered by inclusion.

By equivariant Darboux, every $X \in \mathcal{X}$ is itself a symplectic T invariant manifold with moment map $\phi|_X$. Therefore, by the Atiyah-Guillemin-Sternberg convexity theorem [At] [GS], $\phi(X) \subset \mathfrak{t}^*$ is a convex polytope.

Definition 2.1. *Let a torus T act on a compact symplectic manifold (M, ω) with moment map $\phi : M \rightarrow \mathfrak{t}^*$. The **x-ray** of (M, ω, ϕ) is the closed orbit type stratification \mathcal{X} of M , and the convex polytope $\phi(X)$ for each $X \in \mathcal{X}$.*

Notice that we do not need to understand the geometry of each X ; \mathcal{X} simply functions as a (partially ordered) index set. More abstractly, **an x-ray** is any partially ordered set with a convex polytope associated to each element.

Example 2.2. Define a manifold

$$\tilde{M} = \left\{ [x_0, x_1, x_2, y_0, y_1, y_2] \in \mathbb{CP}^5 \mid x_i y_j^4 = x_j y_i^4 \text{ for all } i \text{ and } j \right\}.$$

Let $\tilde{\omega}$ be the Fubini-Study symplectic form. The torus $S^1 \times S^1$ acts on \tilde{M} by

$$(\alpha, \beta) \cdot [x_0, x_1, x_2, y_0, y_1, y_2] = [x_0, \alpha^4 x_1, \beta^4 x_2, \alpha \beta y_0, \alpha^2 \beta y_1, \alpha \beta^2 y_2].$$

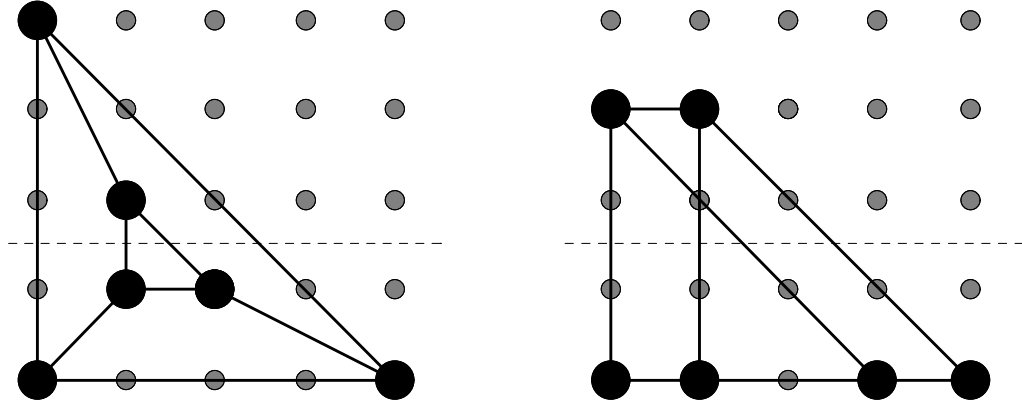


FIGURE 1. The x-rays for Example 2.2 and Example 2.3

The x-ray for a moment map $\tilde{\phi} : M \rightarrow \mathbb{R}^2$ appears in the left hand side of Figure 1.¹

The small grey dots denote the weight lattice; the dot in the bottom left is the origin. Each large black dot is the image of a connected component of the fixed point set. Each black line is the image of a connected component of the set of points with a given one dimensional stabilizer. The dashed line will be explained later.

Example 2.3. Let $S^1 \times S^1$ act on $\hat{M} = \mathbb{CP}^1 \times \mathbb{CP}^2$ by

$$(\alpha, \beta) \cdot ([x_0, x_1], [y_0, y_1, y_2]) = ([\alpha x_0, x_1], [\alpha y_0, \beta y_1, y_2]).$$

Let $\hat{\omega} = \pi_1^*(\omega_1) + 3*\pi_2^*(\omega_2)$, where π_i is the projection map onto the i 'th componet, and ω_i is the Fubini-Study symplectic form on \mathbb{CP}^i . The x-ray for a moment map $\hat{\phi} : M \rightarrow \mathbb{R}^2$ appears in the right hand side of Figure 1.²

3. THE EXTENSION CRITERION

In this section, we define the etension criterion. We then prove that if the x-ray of a symplectic manifold with a Hamiltonian torus action does not satisfy this criterion, then the symplectic manifold does not admit a compatible invariant complex structure.

A convex polytope $\Delta \subset \mathfrak{t}^*$ is **compatible** with an x-ray (\mathcal{X}, ϕ) if

1. for each face σ of Δ , there exists $X_\sigma \in \mathcal{X}$ such that $\dim(\phi(X_\sigma)) = \dim(\sigma)$ and $\sigma \subseteq \phi(X_\sigma)$;
2. if σ and σ' are faces of Δ such that $\sigma \subset \sigma'$, then $X_\sigma \subset X_{\sigma'}$.

Let a torus T act on a symplectic manifold (M, ω) with moment map $\phi : M \rightarrow \mathfrak{t}^*$. If $N \subset M$ is a symplectic T -invariant submanifold, then the convex polytope $\phi(N)$ is compatible with the x-ray of (M, ω, ϕ) . There are exactly five two dimensional (and nine one dimensional) convex polytopes which are compatible with the x-rays in Examples 2.2 and 2.3.

Similarly, a convex cone $C \subset \mathfrak{t}^*$ is **compatible** with an x-ray \mathcal{X} if there is a neighborhood U of the vertex of C such that

¹Indeed, a three dimensional torus acts effectively on \tilde{M} , that is \tilde{M} is a symplectic toric manifold. The action described above is the action of a two dimensional subtorus. The above x-ray is thus the projection of a three dimensional polytope.

²The footnote above also applies to \hat{M} .

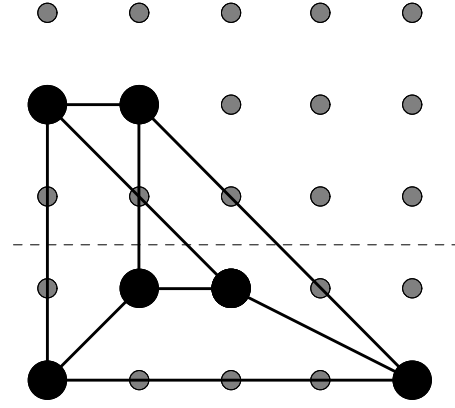


FIGURE 2. An x-ray which does not satisfy the extension criterion

1. for each face σ of C , there exists $X_\sigma \in \mathcal{X}$ such that $\dim(\phi(X_\sigma)) = \dim(\sigma)$ and $\sigma \cap U \subseteq \phi(X_\sigma)$, and
2. if σ and σ' are faces of C such that $\sigma \subset \sigma'$, then $X_\sigma \subset X_{\sigma'}$.

A convex polytope Δ is an **extension** of a convex cone C if there is a neighborhood U of its vertex such that $C \cap U = \Delta \cap U$.

Definition 3.1. An x-ray satisfies the **extension criterion** if every compatible strictly convex cone can be extended to a compatible convex polytope.

The x-rays in Figure 1 satisfy the extension criterion.

Example 3.2. The x-ray in Figure 2 does not satisfy the extension criterion. For instance, the cone

$$\{(s, t) \in \mathbb{R}^2 \mid t \geq 1 \text{ and } s + t \leq 3\}$$

is compatible with the x-ray, but it does not extend to a compatible polytope. (Remember that the origin is in the bottom left corner). Notice, however, that below the dashed line this x-ray resembles the x-ray on the left in Figure 1, and that above the dashed line this x-ray resembles the x-ray on the right in Figure 1.

Theorem 3.3. Let a torus T act on a compact symplectic manifold (M, ω) with moment map $\phi : M \rightarrow \mathfrak{t}^*$. If the x-ray of (M, ω, ϕ) does not satisfy the extension criterion (see Definition 3.1), then (M, ω) does not admit a compatible T -invariant complex structure.

Proof. This theorem follows directly from Theorem 3.4, which is a reformulation of Theorem 2 in [At], and Lemma 3.5. \square

The converse need not be true.

Theorem 3.4. (Atiyah) Let a torus T act by holomorphic symplectomorphisms on a compact Kähler manifold (M, ω, J) with moment map $\phi : M \rightarrow \mathfrak{t}^*$. Then $T_{\mathbb{C}}$, the complexification of T , also acts on M . For $y \in M$, the set $\phi(\overline{T_{\mathbb{C}} \cdot y})$ is a convex polytope which is compatible with the x-ray of (M, ω, ϕ) , where $\overline{T_{\mathbb{C}} \cdot y}$ is the closure of the $T_{\mathbb{C}}$ orbit of y .

Remark. In the case we need, the action of T on M is quasi-free. Therefore, $\overline{T_{\mathbb{C}} \cdot y}$ is a smooth symplectic manifold, and $\phi(\overline{T_{\mathbb{C}} \cdot y})$ is a convex polytope by the Atiyah-Guillemin-Sternberg convexity theorem [At] [GS].

Lemma 3.5. *Let a torus T act by holomorphic symplectomorphisms on a compact Kähler manifold (M, ω, J) with moment map $\phi : M \rightarrow \mathfrak{t}^*$. Let C be a strictly convex cone which is compatible with the x-ray of (M, ω, ϕ) . There exist $y \in M$ such that $\phi(\overline{T_{\mathbb{C}} \cdot y})$ is an extension of C , where $\overline{T_{\mathbb{C}} \cdot y}$ is the closure of the $T_{\mathbb{C}}$ orbit of y , and $T_{\mathbb{C}}$ is the complexification of T .*

Proof. Since C is compatible with the x-ray of (M, ω, ϕ) , there is a fixed point component $F \subset M$ which corresponds to the vertex of C . Given $m \in F$, let $\xi_1, \dots, \xi_n \in \mathfrak{t}^*$ be the weights for the action of $T_{\mathbb{C}}$ on $T_m M$. Let $T_{\mathbb{C}}$ act on \mathbb{C}^n by $t \cdot (z_1, \dots, z_n) = (t^{\xi_1} z_1, \dots, t^{\xi_n} z_n)$. There exists a $T_{\mathbb{C}}$ equivariant holomorphic diffeomorphism ψ from a neighborhood of 0 in \mathbb{C}^n to a neighborhood of m in M . (see [S]).

Define $\omega' := \psi^*(\omega)$ and $\phi' := \psi^*(\phi)$. Given $J \subset \{1, \dots, n\}$, let $E_J = \{z \in \mathbb{C}^n \mid z_i \neq 0 \text{ if and only if } i \in J\}$. The image $C_J := \phi'(\overline{E_J})$ is equal to $\{\sum_{j \in J} a_j \xi_j \mid a_j \geq 0\}$. Since C is compatible with the x-ray of (M, ω, ϕ) it is also compatible with the x-ray of $(\mathbb{C}^n, \omega', \phi')$. Therefore, $C = C_J$ for some unique minimal J .

Choose $y' \in E_J$. On the one hand, $\phi'(\overline{T_{\mathbb{C}} \cdot y'}) \subset \phi'(\overline{E_J}) = C_J$. On the other hand, for every $j \in J$, $C_{\{j\}}$ is a proper face of C_J . Therefore, there exists $\eta \in \mathfrak{t}$ such that $\langle \eta, \xi_j \rangle = 0$ but $\langle \eta, \xi_i \rangle < 0$ for every $i \in J$ such that $i \neq j$. It follows that $E_{\{j\}} \subset \overline{T_{\mathbb{C}} \cdot y'}$, and hence $C_{\{j\}} \subset \phi'(\overline{T_{\mathbb{C}} \cdot y'})$, for all $j \in J$. Since $\phi'(\overline{T_{\mathbb{C}} \cdot y'})$ is convex, $\phi'(\overline{T_{\mathbb{C}} \cdot y'}) = C_J = C$.

Finally, define $y := \psi^{-1}(y')$. Since $\phi'^{-1}(\phi'(0)) = 0$, there exists a neighborhood U of the vertex of C such that $\phi_{T_{\mathbb{C}} \cdot y} \cap U = C \cap U$. \square

4. CONSTRUCTING AN EXAMPLE

In this section, we glue together pieces of two Kähler manifolds to construct a symplectic manifold with a Hamiltonian torus action. Because its x-ray does not satisfy the extension criterion, it does not admit an invariant compatible complex structure.

Lemma 4.1. *There exists a compact six dimensional manifold (M, ω) and a two dimensional torus T which acts effectively in a Hamiltonian fashion on M such that (M, ω) does not admit any compatible invariant complex structure. Additionally,*

1. *the manifold M is simply connected,*
2. *the stabilizer of every point is connected,*
3. *all the fixed points are isolated, and*
4. *all the reduced spaces are symplectomorphic to \mathbb{CP}^1 . In particular, they all admit compatible complex structures.*

Proof. The basic idea of the construction is very simple. Define $(\tilde{M}, \tilde{\omega}, \tilde{\phi})$ and $(\hat{M}, \hat{\omega}, \hat{\phi})$ as in Example 2.2 and Example 2.3, respectively. We show that there exists a neighborhood $W \subset \mathbb{R}^2$ of the dashed line in these examples such that $\tilde{\phi}^{-1}(W)$ and $\hat{\phi}^{-1}(W)$ are isomorphic. We define a new symplectic manifold by gluing the “bottom” of \tilde{M} to the “top” of \hat{M} .³

The first step is to show that if we reduce both spaces along the dashed line $D = \{(s, t) \in \mathbb{R}^2 \mid t = 1.5\}$, the resulting spaces are isomorphic. More precisely, define $H := e \times S^1 \subset S^1 \times S^1$. The moment map for the H action is the projection of the $S^1 \times S^1$ moment map onto its second component. Moreover, H acts freely on $\tilde{\phi}^{-1}(D)$ and $\hat{\phi}^{-1}(D)$. Therefore, the reduced spaces $\tilde{M}/H = \tilde{\phi}^{-1}(D)/H$ and

³Alternatively, this manifold can be constructed using Gompf gluing.

$\hat{M}/H = \hat{\phi}^{-1}(D)/H$ are smooth symplectic manifolds on which T/H acts effectively in a Hamiltonian fashion.

The T/H action on both \tilde{M}/H and \hat{M}/H , is free except for at four isolated fixed points. Moreover, the image of these four points under the T/H moment map is the same for both reduced spaces. Applying the classification of four dimensional manifolds with a Hamiltonian circle action given by Y. Karshon, the reduced spaces are equivariantly symplectomorphic [Ka]. (See also [Au] and [AH]).

Alternatively, \tilde{M} and \hat{M} are symplectic toric manifolds with associated three dimensional polytopes. The reduced spaces are the symplectic toric manifolds which are associated with the cross section of these polytopes. Specifically, \tilde{M}/H is equivariantly symplectomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$ with the diagonal action and \hat{M}/H is equivariantly symplectomorphic to the Hirzebruch surface $\{[x, y, w, z] \in \mathbb{CP}^3 \mid xy^3 = wz^3\}$ with the S^1 action given by $\lambda \cdot [x, y, w, z] = [x, \lambda^3 y, \lambda w, \lambda^2 z]$. (The symplectic form is not the Fubini-Study form in either case.) It is straightforward to write down explicitly an S^1 equivariant symplectomorphism between these two spaces⁴.

Next, we show that there exists an equivariant diffeomorphism from $\tilde{\phi}^{-1}(D)$ to $\hat{\phi}^{-1}(D)$ such that $\tilde{\omega}$ is the pull back of $\hat{\omega}$. Because \tilde{M}/H and \hat{M}/H are equivariantly symplectomorphic, it is enough to show that $\tilde{\phi}^{-1}(D)$ and $\hat{\phi}^{-1}(D)$ are isomorphic as T/H equivariant line bundles.

For any equivariant line bundle over $\tilde{M}/H = \hat{M}/H = \mathbb{CP}^1 \times \mathbb{CP}^1$, there is a representation of T/H on the fiber of each fixed point in the base manifold. The Chern class of the line bundle is determined once we know the representation. (For example, the integral of the Chern class over the first sphere is given by the difference of the representations at (N, N) and (S, N) , where S and N denote the north and south poles, respectively.)

Moreover, these fixed-point representations are determined by their kernels. Their kernels, in turn, are determined by the stabilizers of points in $\tilde{\phi}^{-1}(D)$ and $\hat{\phi}^{-1}(D)$. Since all stabilizers are connected, they can be read from the x-ray. Since \tilde{M} and \hat{M} have identical x-rays near D , they have the same representations at corresponding fixed points, therefore there exists an equivariant diffeomorphism from $\tilde{\phi}^{-1}(D)$ to $\hat{\phi}^{-1}(D)$ such that $\tilde{\omega}$ is the pull back of $\hat{\omega}$.

By the equivariant coisotropic embedding theorem there exists neighborhoods of $\tilde{\phi}^{-1}(D)$ and $\hat{\phi}^{-1}(D)$ which are equivariantly symplectomorphic. Because \tilde{M} and \hat{M} are compact there exists a neighborhood W of D such that there exists a equivariant symplectomorphism $f : \tilde{\phi}^{-1}(W) \rightarrow \hat{\phi}^{-1}(W)$.

Define open sets $U := W \cup \{(s, t) \in \mathbb{R}^2 \mid t < 1.5\}$ and $V := W \cup \{(s, t) \in \mathbb{R}^2 \mid t > 1.5\}$. We define a new compact symplectic manifold (M, ω) with an effective Hamiltonian T action by

$$M := \tilde{\phi}^{-1}(U) \amalg \hat{\phi}^{-1}(V) / \sim ,$$

where $x \sim f(x)$ for all $x \in \tilde{\phi}^{-1}(W)$.

Over U , the x-rays of M and \tilde{M} agree; over V , the x-rays of M and \hat{M} agree. The x-ray of (M, ω, ϕ) is the one drawn in Example 3.2, which does not satisfy the extension criterion. By Theorem 3.3, (M, ω) does not admit a compatible T invariant complex structure.

To see that M is simply connected, it is enough to check that the H moment map is a Morse function and that the index of every critical point is even.

⁴In contrast, these two spaces are not holomorphically equivalent. Therefore, the complex structures on \tilde{M} and \hat{M} will not glue together to create a complex structure on M .

The other claims follow from the construction. \square

5. VARYING THE SYMPLECTIC FORM

In this section, we show that the manifold which we constructed in Section 4 does not admit any T -invariant Kähler structure.

Let a torus T act effectively on a compact manifold M . There may be many invariant symplectic forms ω on M , each with its own moment map $\phi : M \rightarrow \mathfrak{t}^*$. However, the x-ray of (M, ω, ϕ) only depends in a limited way on the symplectic form ω . The closed orbit type stratification \mathcal{X} depends *only* on the action of T on M . In contrast, the polytope $\phi(X) \subset \mathfrak{t}^*$ for $X \in \mathcal{X}$ does depend on the cohomology class of ω (though not on ω itself). By the Atiyah-Guillemin-Sternberg convexity theorem [At] [GS], $\phi(X)$ is the convex hull of $\{\phi(F) \mid F \subset X \text{ and } F \in \mathcal{X}_{\{1\}}\}$, where $\mathcal{X}_{\{1\}}$ denotes the set of components of the fixed point set. Therefore, the x-ray is determined by $\phi(F)$ for $F \in \mathcal{X}_{\{1\}}$. In fact, given F and F' in $\mathcal{X}_{\{1\}}$ and $X \in \mathcal{X}_K$ such that $F \subset X$ and $F' \subset X$, $\phi(F)$ and $\phi(F')$ must satisfy $\phi(F) - \phi(F') \in \mathfrak{k}^\circ \subset \mathfrak{t}^*$, where \mathfrak{k}° denotes the annihilator of the Lie algebra of K . Therefore, the x-ray is determined by even less information.

Theorem 5.1. *There exists a compact six dimensional symplectic manifold (M, ω) and a two dimensional torus T which acts effectively on M in a Hamiltonian fashion such that M does not admit any invariant Kähler structure. Additionally,*

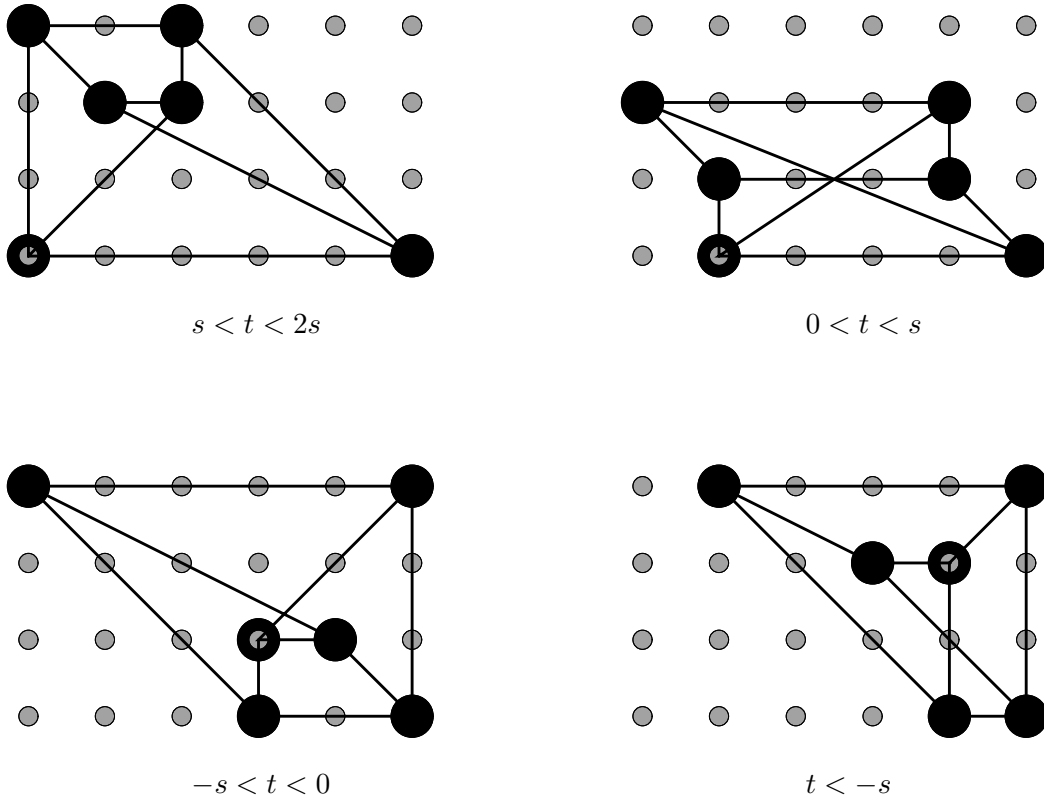
1. *the manifold M is simply connected,*
2. *the stabilizer of every point is connected,*
3. *all the fixed points are isolated, and*
4. *all the reduced spaces are symplectomorphic to \mathbb{CP}^1 . In particular, they all admit Kähler structures.*

Proof. Let (M, ω, ϕ) be the symplectic manifold constructed in Lemma 4.1. Let F and F' denote the components of the fixed point set such that $\phi(F) = (0, 0)$ and $\phi(F') = (1, 3)$.

Now choose *any* $S^1 \times S^1$ invariant symplectic form ω' on M . Since M is simply connected, there is a moment map $\phi' : M \rightarrow \mathbb{R}^2$. To prove the theorem, we must show that the x-ray of (M, ω', ϕ') fails to satisfy the extension criterion.

Applying the discussion in the previous paragraph, this x-ray is determined by $\phi'(F)$ and $\phi'(F')$. Moreover, since the extension criterion is not affected by translations, we may assume that $\phi'(F)$ is zero. Therefore, the x-ray is determined by $\phi'(F') = (s, t)$. In particular, the components of the fixed point set which are mapped by ϕ to $(0, 0), (4, 0), (1, 1), (2, 1), (0, 3)$, and $(1, 3)$, will be mapped by ϕ' to $(0, 0), (s+t, 0), (s, s)(t-s, s), (0, t)$ and (s, t) , respectively. We may further restrict attention to the case $s \geq 0$ because the x-rays for $\phi'(F') = (s, t)$ and $\phi'(F') = (-s, -t)$ are mirror images.

Notice that the extension criterion does not depend on the length of the edges; it can only change as one of the edges “switches direction”. One or more of the edges will have zero length exactly if $s = 0, t = 0, t = -s, t = s$, or $t = 2s$. This leaves us with five cases: $0 < t < s$, $s < t < 2s$, $0 < t < s$, $-s < t < 0$, and $t < -s$. The first case has already been discussed. Representative samples of the other cases appear in Figure 3; $\phi'(F) = (0, 0)$ is denoted by a grey dot in a large black dot. Note that $0 < t < s$ is impossible because the overall shape is not convex. The other x-rays may be possible but do not satisfy the extension property. \square

FIGURE 3. Possible x-rays for other symplectic forms on M

Remark. Because every odd Betti number is zero, we cannot prove that M is not Kähler using that route. Additionally, as C. Woodward has pointed out, M does satisfy the hard Lefschetz property. At the time of this writing, it is not known whether or not M admits a Kähler structure.

6. MORE EXAMPLES

In this section, we describe how to construct many more symplectic manifolds with Hamiltonian torus actions which do not admit compatible invariant complex structures. These profuse examples show that the category of symplectic manifolds with Hamiltonian torus actions is much richer than the category of Kähler manifolds with compatible Hamiltonian torus actions.

The construction in Section 4 can be extended to create other symplectic manifolds with Hamiltonian torus actions which do not admit compatible invariant complex structures. Instead of beginning with Example 2.2 and 2.3, let a two dimensional torus T act in a Hamiltonian fashion on any two six dimensional symplectic manifolds, which we'll denote by $(\tilde{M}, \tilde{\omega})$ and $(\hat{M}, \hat{\omega})$. Consider a subtorus $H \subset T$ and $\eta \in \mathfrak{t}^*$. Define $D := \{\alpha \in \mathfrak{t}^* | \pi(\alpha) = \eta\}$, where $\pi : \mathfrak{t}^* \rightarrow \mathfrak{h}^*$ is the projection map. The construction in Lemma 4.1 can be applied to create a new manifold when the following conditions are satisfied: The reduction of \tilde{M} by H at η and the reduction of \hat{M} by H at η are smooth symplectic manifolds. The circle T/H acts on both reduced spaces with isolated fixed points. The x-rays of $(\tilde{M}, \tilde{\omega}, \tilde{\phi})$ and $(\hat{M}, \hat{\omega}, \hat{\phi})$ are identical near D , and the corresponding elements of the closed orbit type stratifications have the same stabilizer subgroups. With a little patience, it is possible to construct an abundance of manifolds this way. Typically, they do not satisfy the

extension criterion. Moreover, the arguments in Section 5 can be adapted to show that many of them do not admit any invariant Kähler structure.

Another way to create examples is to start with a symplectic manifold (M, ω) with a Hamiltonian torus action which does not admit an invariant Kähler structure. The product of (M, ω) and any symplectic manifold on which a torus acts with isolated fixed points will have M itself as a closed orbit type strata. Therefore, it cannot admit an invariant Kähler structure.

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